Question 1. 40 in total

$$\text{Data } X = \begin{cases} 2 & \text{w.p. } \theta \\ 4 & \text{w.p. } \theta^2 \\ 6 & \text{w.p. } 1-\theta-\theta^2 \end{cases} \quad (0 \le \theta \le .618)$$

5 (a) Show that X is complete 3 as well as (trivially) sufficient. 2

5 (b) Find with explanation the UMVUE for θ and θ^2.

Suppose now your data was a random sample of size 2, (X_1, X_2) from the above distribution

5 (c) Find the joint distribution of (X_1, X_2).

7 (d) Show that the pair (X_1, X_2) is not complete, and the sum $T = X_1 + X_2$ is not sufficient

8 (e) Write an algebraic expression for the "little" Fisher information $i(\theta)$ in X (or X_1).

5 (f) Does your UMVUE for θ, based on the single observation X above, achieve the Cramér-Rao bound?

5 (g) If the support point "6" in the distribution of X is replaced by "8", what happens to the sufficiency result in part (d)? Explain.

(h)

60!

Take-home question (Open books, notes, computers, etc.)

3 (a) Explain what is meant by a location family $\mathcal{F} = \{f_\theta\}$ with location parameter $\theta \in \mathbb{R}$, and with "generating shape" f_0.

6 (b) Derive a formula $I(\theta)$ for the Fisher information of such a family. (The answer involves f_0, but is free of θ.)

(c) Tukey's biweight, $f_0(x)$ is a nice differentiable density supported on $[-1, 1]$ given by

$$f_0(x) = \frac{15}{16}(1-x^2)^2 \, I[|x| \le 1]$$

6 (d) Compute the variance σ_0^2 of $X \sim f_0$

3 (e) On the same axes sketch f_0 and φ, the standard normal density. Your sketch should roughly capture the relative dispersions, i.e. the sizes of the two standard deviations.

6 (f) Compute the Fisher information $I(\theta)$ of the location family $\mathcal{F}$ generated by Tukey's biweight f_0. Compare it to the Fisher information of the location family $\mathcal{F}_N = \{N(\theta, \sigma_0^2) : \theta \in \mathbb{R}\}$ (σ_0^2 being given by part (d)).

3 (g) In which of these families, $\mathcal{F}$ and $\mathcal{F}_N$ is efficient estimation of θ "easier", in terms of the Cramér-Rao

bound for the sampling variance of an unbiased estimator

3 (h) How do we know that with iid-sampling the sample mean is unbiased for both of these families?

3 (i) If the sample sizes are large how do the performances of the sample mean compare for estimating θ in $\mathcal{F}$ and $\mathcal{F}_W$ (In 281B we learn how to assess large-sample performance of medians as well.)

6 (j) Do you think the sample mean is UMVU for estimating θ in $\mathcal{F}$? Explain.

3 (k) Explain why the order statistics are sufficient in iid. sampling from any family of distributions on the line.

6 (l) Suppose I tell you the order statistics are minimal sufficient when sampling from $\mathcal{F}$. Explain how you know then that $\mathcal{F}$ is not an exponential family. (You cannot just point to the density formula and say it doesn't "look" as if it can be written in exponential-family form.)

6 (m) Are the order statistics complete when sampling from $\mathcal{F}$? Explain.

in the parameter space

~~in the parametric space~~

6 (n) If $X \sim f_\theta \in \mathcal{F}$, where is the bias of e^X for e^θ upwards, and where is it downwards? Explain.