Midterm

MATH 281A

November 23, 2014

Problem 1.

1. Sufficiency means that the distribution of X conditional on $\delta(X)$ has no relationship with θ. You may use factorial theorem here since it is an equivalent statement. Completeness means that if you have a measurable function g such that $\mathbb{E}_\theta(g(\delta(X))) = 0$ for any θ in the natural parameter space, then $g = 0$ almost surely.

2. The density for such a distribution family is

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}2\mu} \exp\{-\frac{(x-\mu)^2}{8\mu^2}\}$$

which is equivalent to

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}2\mu} \exp\{-\frac{x^2}{8\mu^2} + \frac{x}{4\mu} - \frac{1}{8}\}$$

Clearly this is an exponential family.

3. Since the sample is iid, we simply multiply the density and this gives the joint density as

$$f_\mu(x_1, x_2, \ldots, x_n) = (\frac{1}{\sqrt{2\pi}2\mu})^n \exp\{-\frac{\sum_{i=1}^n x_i^2}{8\mu^2} + \frac{\sum_{i=1}^n x_i}{4\mu} - \frac{n}{8}\}$$

Here we have used the result of (2).

4. Factorization theorem show that $(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2)$ is a sufficient statistic.

5. The pair $(-\frac{1}{8\mu^2}, \frac{1}{4\mu})$ can not fulfill an open set/open ball/square in $\mathbb{R}^2$. Therefore the theorem fails.

6. Consider estimating μ^2 by $\bar{X}^2$ and $\sum_{i=1}^n x_i^2$. Since we have $\bar{X}$ follows $N(\mu, 4\mu^2/n)$, we have

$$\mathbb{E}_\mu(\bar{X}^2) = \mu^2 + 4\mu^2/n = \frac{n+4}{n}\mu^2$$

By the formula of variance we have

$$\mathbb{E}_\mu(X_1^2) = \mu^2 + 4\mu^2 = 5\mu^2$$

Therefore

$$\mathbb{E}_\mu(\frac{1}{5n}\sum_{i=1}^n X_i^2 - \frac{n}{n+4}\bar{X}^2) = 0$$

for any μ. So the statistic is not complete.

Problem 2.

For Poisson distribution we know that $\bar{X}$ is sufficient and complete. So the conditional expectation $\mathbb{E}((X_1 - X_2)^2|\bar{X})$ is the UMVUE for $\mathbb{E}_\lambda((X_1 - X_2)^2)$ by Rao-Blackwell.

By calculation we have

$$\mathbb{E}_\lambda((X_1 - X_2)^2) = \mathbb{E}_\lambda(X_1^2 + X_2^2 - 2X_1X_2) = 2(\lambda + \lambda^2) - 2\lambda^2 = 2\lambda$$

By observation we know that $2\bar{X}$ is an unbiased estimator for 2λ and is purely based on a sufficient and complete statistic. Therefore by Lehmann-Scheffe we know that $2\bar{X}$ is also a UMVUE for 2λ. By the uniqueness of the UMVUE we know that $\mathbb{E}_\lambda((X_1 - X_2)^2) = 2\bar{X}$

Problem 3.

1. For a single sample, we have

$$\log f_{\sigma^2}(x) = -\log(\sqrt{2\pi}) - \log(\sigma) - \frac{x^2}{2\sigma^2}$$

Take twice derivative against σ^2 we know

$$\frac{\partial^2 \log f_{\sigma^2}(x)}{\partial(\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{x^2}{\sigma^6}$$

So the Fisher information for X_1 is just

$$i(\sigma^2) = -\mathbb{E}_{\sigma^2}(\frac{\partial^2 \log f_{\sigma^2}(X)}{\partial(\sigma^2)^2}) = \frac{1}{2\sigma^4}$$

2. The complete and sufficient statistic for the parameter is $\sum_{i=1}^n X_i^2$, however this statistic is not based on that. Therefore it is not UMVUE.

3. Using Rao-Blackwell, $\mathbb{E}(X_1^{12}/10395|\sum_{i=1}^n X_i^2)$ is the UMVUE.

4. The Creamer-Rao bound is calculated by

$$\frac{(g'(\sigma^2))^2}{I(\sigma^2)} = \frac{(6(\sigma^2)^5)^2}{ni(\sigma^2)} = \frac{72\sigma^{24}}{n}$$