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UCSD Student Number: Reference Answer Sheet **MATH 281C (Spring 2015)**

Answer the question in the space provided. Provide concise and precise answers (vagueness will be penalized). Make sure the presentation is neat (easy to read). Define any notation that you introduce. Name any result you use.

Problem 1. Below $u : \mathbb{R} \mapsto \mathbb{R}$ is measurable. The underlying measure is Lebesgue's.

1. Suppose $\int_0^1 u(x)\mathrm{d}x = 0$ and $\sup_x |u(x)| < \infty$. Show that $p_\theta(x) = 1 + \theta u(x)$ defines a family of probability distributions when θ ranges in some interval (define that interval explicitly). Show it is QMD at $\theta = 0$, compute the quadratic mean derivative η_0 and information $I(0)$.

(1) $p_\theta(x) \geq 0$ gives $-\sup_x u(x) \leq \theta \leq -\inf_x u(x)$

(2) Consider $\eta_0 = \frac{1}{2}u(x)$, we get

$$\int_0^1 [\sqrt{p_{0+h}(x)} - \sqrt{p_0(x)} - h\eta_0]^2 dx = \int_0^1 (\sqrt{1+hu(x)} - 1 - h\eta_0)^2 dx$$

$$= \int_0^1 (1 + \tfrac{1}{2}u(x)h + o(|h|) - 1 - h\cdot\tfrac{1}{2}u(x))^2 dx = o(|h|^2).$$

So it's QMD at $\theta=0$ and $\eta_0 = \frac{1}{2}u(x)$.

(3) The Fisher information is

$$I(0) = 4\int \eta_0^2(x)\, dx = \int u^2(x)\, dx.$$

2. Suppose $\int_0^1 e^{\theta u(x)}\mathrm{d}x < \infty$ when $|\theta| < \varepsilon$ for some $\varepsilon > 0$. Consider $p_\theta(x) = C(\theta)\exp(\theta u(x))$. Explicitly define $C(\theta)$ so that this is a density. Show this family, as $\theta \in (-\varepsilon, \varepsilon)$, is QMD at $\theta = 0$, compute the quadratic mean derivative η_0 and information $I(0)$.

(1). $C(\theta) = \left(\int_0^1 e^{\theta u(x)} dx\right)^{-1}$ to make sure $\int p_\theta(x)\, dx = 1$.

(2). Consider $\eta_0 = \frac{1}{2}u(x)$, we get

$$\int_0^1 [\sqrt{p_{0+h}(x)} - \sqrt{p_0(x)} - h\eta_0]^2 dx = \int_0^1 (c^{\frac{1}{2}}(h)\, e^{\frac{1}{2}hu(x)} - 1 - h\eta_0)^2 dx$$

$$= \int_0^1 (c^{\frac{1}{2}}(h) - 1)^2 (1 + \tfrac{1}{2}u(x))\, dx + o(h^2) \quad \text{(Taylor expansion)}$$

Since $\left.\frac{\partial C^{\frac{1}{2}}(\theta)}{\partial\theta}\right|_{\theta=0} = 0$, $C^{\frac{1}{2}}(h) - 1 = o(h)$. ~~so $\int_0^1 (C^{\frac{1}{2}}(h)-1)^2 dx = o(h)$~~

so $\int_0^1 (C^{\frac{1}{2}}(h) - 1)^2 (1 + \frac{1}{2}u(x))\, dx = o(h^2)$.

(3) $\eta_0 = \frac{1}{2}u(x)$, so $I(0) = \int u^2(x)\, dx$.

Problem 2. Let P_θ be the uniform distribution on $(0, \theta)$. For what values of $a \in \mathbb{R}$ are P_1^n and P_{1+h/n^a}^n (mutually) contiguous? Consider the cases where $h < 0$ and $h > 0$. Prove your assertions.

~~By symmetry we only need to show~~ $\cancel{h<0}$ ~~and~~ $\cancel{P^n_{1+h/n^a}}$ ~~contiguous to~~ P_1^n

By symmetry we only need to show $h>0$ and P_1^n contiguous to P^n_{1+h/n^a}

① P_1^n is absolutely continuous to P^n_{1+h/n^a}, so for E_n, $P_1^n(E_n) \leq (1+\frac{h}{n^a})^n P^n_{1+h/n^a}(E_n)$

as $a>1$, $(1+\frac{h}{n^a})^n \to 1$. As $P_{1+h/n^a}(E_n) \to 0$, $P_1^n(E_n) \to 0$.

(2). For E_n such that $P_1^n(E_n) \to 0$, ~~devide~~ divide $E_n = E_n \cap \text{supp}(P_1^n) + E_n \setminus \text{Supp}(P_1^n)$.

and $P_{1+h/n^a}(E_n \cap \text{supp}(P_1^n)) \leq (1+\frac{h}{n^a})^{-n} P_1(E_n) \leq P_1(E_n)$

$P_{1+h/n^a}(E_n \setminus \text{Supp}(P_1^n)) \leq [\frac{h/n^a}{1+h/n^a}]^n = [1+\frac{n^a}{h}]^{-n}$

as $a>1$. $[1+\frac{n^a}{h}]^{-n} \to 0$.

Therefore $a>1$ makes them contiguous. ($a>1$ is also a necessary condition.)

Problem 3. Let P_n and Q_n be probability distributions on the same measurable space. Let p_n and q_n be their respective densities with respect to a common dominating measure. Let $L_n(x) = q_n(x)/p_n(x)$ if $p_n(x) > 0$, and $L_n(x) = \infty$ otherwise. Suppose that Q_n is contiguous to P_n. Show that $\mathbb{E}_{P_n}(L_n) \to 1$. [HINT: Recall that $P_n(\{x : p_n(x) > 0\}) = 1$.]

$$E_{P_n}(L_n) = \int L_n(x)\, dP_n(x) = \int \frac{q_n(x)}{p_n(x)} \cdot p_n(x)\, dx$$

$$= \int_{\{x:\, p_n(x)>0\}} \frac{q_n(x)}{p_n(x)} \cdot p_n(x)\, dx = \int_{\{x:\, p_n(x)>0\}} q_n(x)\, dx$$

Since $P_n(\{x: p_n(x)>0\}) = 1$, and Q_n contiguous to P_n,

we have $Q_n(\{x: p_n(x) = 0\}) \to 0$

which is just $\int_{\{x:\, p_n(x)>0\}} q_n(x)\, dx \to 1$.

Problem 4. Consider a density f with respect to the Lebesgue measure on $\mathbb{R}$ that has mean zero and unit variance, is differentiable and such that the location family $\{f(\cdot - \theta) : \theta \in \mathbb{R}\}$ is QMD. We want to test $\theta = 0$ versus $\theta > 0$ based on an IID sample $X_1, \ldots, X_n$ from $f(\cdot - \theta)$. Form the z-ratio $Z_n = \sqrt{n}\bar{X}_n$, where $\bar{X}_n$ is the sample mean.

1. Derive the asymptotic distribution of Z_n under $\theta = 0$. Use this to define a test based on Z_n with asymptotic level $\alpha \in (0, 1)$ (which is given).

By central limit theorem (since the second moment is finite),

we have $\sqrt{n}\,\bar{X}_n \xrightarrow[\theta=0]{w} N(0,1)$.

Then the test is naturally

$$T(X) = \begin{cases} 1 & \sqrt{n}\,\bar{X}_n > Z_{1-\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

where $Z_{1-\alpha}$ is the $1-\alpha$ quantile of a standard normal.

2. Derive the asymptotic distribution of Z_n under $\theta = h/\sqrt{n}$ where $h > 0$ is fixed. What is the asymptotic power of the test (defined above) against this alternative?

By method of Example 12.3.9

$$\sigma_{1,2} = -\frac{h}{\sigma}\,\mathrm{cov}_{\theta=0}\left(X_i, \frac{f'(X_i)}{f(X_i)}\right) = -\frac{h}{\sigma}\int x f'(x)\,dx$$

$$= \frac{h}{\sigma} = h \quad (\sigma = 1 \text{ in this case}).$$

Then by Cor 12.3.2. $Z_n \xrightarrow[\theta=h/\sqrt{n}]{w} N(h, 1)$

Then $P_{\theta=h/\sqrt{n}}(Z_n > Z_{1-\alpha}) = P_{\theta=h/\sqrt{n}}(Z_n - h > Z_{1-\alpha} - h)$

$$\longrightarrow 1 - \Phi(Z_{1-\alpha} - h) = \Phi(h - Z_{1-\alpha})$$