Homework 3

MATH 281B

February 1, 2015

Exercise Show ARE of sample mean and median for unimodal model $\geq 1/3$.

Without loss of generality we assume the mean and median as 0 and the distribution is symmetry. Then we see that $f(0)$ is the largest value of the density function. The median has an asymptotic variance of $1/(4f(0)^2)$ and mean has variance σ^2. The target is to show that

$$\sigma^2 \geq \frac{1}{3} \cdot \frac{1}{4f(0)^2}$$

or

$$f(0)^2 \int x^2 f(x) dx \geq \frac{1}{12}$$

This can be done by considering another distribution g which is a uniform distribution on $[-1/(2f(0)), 1/2(f(0))]$. First argue that g has a smaller variance of f, and replace the target integration with g.

Exercise Unstability of median.

The target is to show that the asymptotic variance of the sample mean of the distribution $f(x) = |x|$ on $[-1, 1]$ is infinity. We finish this by showing the second moment, multiplied by n, is infinity.

First, for n even, try to show the density of the sample mean is

$$f_n(x) = C_n(\frac{1}{4} - \frac{|x|^4}{4})^{n/2}|x|dx$$

Second, write out the expression of the second moment as

$$n\,\mathbb{E}(|X|^2) = 2n \int_0^1 C_n(\frac{1}{4} - \frac{|x|^4}{4})^{n/2}|x|^3 dx = 2nC_n \int_0^{1/4} (\frac{1}{4} - t)^{n/2} dt = 2\frac{n}{n/2+1} C_n(1/4)^{n/2+1}$$

To hold this finite, we must have $C_n(1/4)^{n/2}$ asymptotically finite. But then the density function will break down to zero. Try to argue this.